A Density Function Connected with a Non-negative Self-decomposable Random Variable

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ABSTRACT

The innovation random variable for a non-negative self-decomposable random variable can have a compound Poisson distribution. In this case, we provide the density function for the compounded variable. When it does not have a compound Poisson representation, there is a straightforward and easily available compound Poisson approximation for which the density function of the compounded variable is also available. These results can be used in the simulation of Ornstein-Uhlenbeck type processes with given marginal distributions. Previously, simulation of such processes uses the inverse of the corresponding tail Lévy measure. We show this approach corresponds to the use of an inverse cdf method of a certain distribution. With knowledge of this distribution and hence density function, the sampling procedure is open to direct sampling methods.

Key words: Infinite divisibility; Ornstein-Uhlenbeck type process; Self-decomposable; Shot noise.

1 Introduction

Recent interest has focused on Ornstein-Uhlenbeck (OU) type processes and their application in stochastic volatility models. This application relies mainly in the subordination of Brownian motion with self-decomposable (SD) processes as operational time. See Sato (2001) for more on subordination using SD processes. A detailed treatment of an application in stochastic volatility models is found in Barndorff-Nielsen and Shephard (2001). An important issue for volatility models is their simulation. OU type processes are stationary processes driven by a positive Lévy process, without Gaussian components, and with SD marginals. Therefore, suitable representations of such processes (similarly SD random variables) can lead to suitable simulation techniques. For a detailed exposition of general Lévy process representations, see Rosiński (2001). A common problem for some of these techniques is the need for the inversion formula of the corresponding tail Lévy measure.

In this paper we discuss the simulation of the innovation random variable associated with SD random variables having tail Lévy measure of the form

$$N((x,\infty)) = \int_x^\infty \frac{1}{y} G(y)\mathrm{d}y$$

for some non-increasing function G. When $G(0)$ is finite, the innovation random variable is compound Poisson and we find the density function of the compounded variable. When $G(0) = \infty$ we use novel approximation methods based on the findings in the finite case in order to

*Email: maprm@bath.ac.uk. The research of the fist author was supported by the National Council for Science and Technology of México (CONACYT) and of the second author by an EPSRC Advanced Research Fellowship.

approximately sample the innovation variable. Our approach relies on the direct sampling of density functions rather than the use of inverse techniques.

The layout of the paper is as follows. In Section 2 we provide a brief background of SD random variables. Section 3 gives the result of the density function of the compounded variable in the case of finite $G(0)$. Section 4 presents a useful equality in distribution which will provide us with an easy way to simulate from the innovation of SD random variables. The result presented in this section is only valid when the tail Lévy measure is finite at zero. However, for the non-finite case, an approximation approach is presented in Section 5. Section 6 provides the relation with OU type processes and their representations.

2 Background

A non-negative random variable X is said to be self-decomposable if for all $0 < \rho < 1$ there exists a (innovation) random variable W_ρ such that

$$X \stackrel{\mathrm{d}}{=} \rho X + W_\rho. \tag{1}$$

Here $\stackrel{\mathrm{d}}{=}$ denotes equal in distribution. Such a random variable X is infinitely divisible (see, for example, Vervaat, 1979) and if $\inf\{x : \mathrm{pr}(X \le x) > 0\} = 0$ (assumed without loss of generality) then X has log-Laplace transform given by

$$\begin{aligned} -\log \mathrm{E}\, e^{-\theta X} &= \int_0^\infty \left(1 - e^{-\theta u}\right) \mathrm{d}N(u) \\ &= \theta \int_0^\infty e^{-\theta y} \overline{N}(y) \mathrm{d}y \end{aligned} \tag{2}$$

where $N(\cdot)$ is a (Lévy) measure on $(0, \infty)$, $\overline{N}(y) = N((y, \infty))$, satisfying $N(1, \infty) < \infty$ and $\int_0^1 u\, \mathrm{d}N(u) < \infty$, or in terms of the tail Lévy measure $\int_0^1 \overline{N}(u)\mathrm{d}u < \infty$. If we denote by $\mathcal{L}_Z(\theta)$ the Laplace transform of the random variable Z, then

$$\begin{aligned} \mathcal{L}_{W_\rho}(\theta) &= \frac{\mathcal{L}_X(\theta)}{\mathcal{L}_X(\rho\theta)} = \exp\left\{-\theta \int_0^\infty (e^{-\theta y} - \rho e^{-\theta \rho y}) \overline{N}(y) \mathrm{d}y\right\} \\ &= \exp\left\{-\theta \int_0^\infty e^{-\theta y} \overline{N}_{W_\rho}(y) \mathrm{d}y\right\}, \end{aligned} \tag{3}$$

where $\overline{N}_{W_\rho}(y) = N((y, y/\rho)) = \overline{N}(y) - \overline{N}(y/\rho)$, is the Lévy measure corresponding to W_ρ and satisfies the same conditions as N.

In the case when X is a non-negative SD random variable, the tail Lévy measure takes the form

$$\overline{N}(x) = \int_x^\infty \frac{1}{y} G(y) \mathrm{d}y,$$

for $x > 0$ and G is a decreasing function. See Sato (1999) Section 53.

If $G(0) := \lim_{y \downarrow 0} G(y) < \infty$, we can set $S(y) := G(y)/G(0)$ as a well-defined survival function corresponding to a distribution function $F(y)$. With this notation, $\overline{N}(x)$ can be rewritten as

$$\overline{N}(x) = \tau \int_x^\infty \frac{1}{y} S(y) \mathrm{d}y. \tag{4}$$

where $\tau = G(0)$.

We will also consider the case when $G(0) = \infty$, to be considered in Section 5. See Bondesson (1982; Section 3) for background and particular examples relating to these sort of Lévy measures. A representation of X as a shot-noise random variable with exponential response is available;

$$X = \sum_{i=1}^{\infty} V_i \, e^{-T_i},$$

where $\{T_i\}$ denotes the sequence of points of a stationary Poisson process with intensity 1 and the V_i are independent and identically distributed from F. See Vervaat (1979) for more details. On the other hand, a representation of X due to Ferguson and Klass (1972), is given by

$$X = \sum_{i=1}^{\infty} J_i,$$

where $N(J_i, \infty) = T_i$.

A problem considered by a number of authors in the early eighties concerned representations of the innovation random variable W_ρ. Suitable representations may lead to simulation methods. Lawrance (1982) found a representation of W_ρ when X is gamma distributed, say $\mathrm{ga}(\tau, 1)$, in terms of a compound Poisson distribution,

$$W_\rho = \sum_{i=1}^{k} Y_i \quad \text{and} \quad k \sim \mathrm{Po}(-\tau \log \rho),$$

where the Y_i are independent and identically distributed random variables. Lawrance (1982) provided the Laplace transform for Y, given by

$$\mathrm{E}\, e^{-\theta Y} = 1 - \frac{\log\left(\frac{1+\rho\theta}{1+\theta}\right)}{\log \rho}$$

and the result that $Y \stackrel{\mathrm{d}}{=} \rho^U E$, where U is a uniform random variable from $[0, 1]$ and E an exponential random variable with mean 1, independent of U. This clearly gives an easy way to simulate random variates from W_ρ.

The main objective of this paper is to generalize the Lawrance (1982) result to a wider family of positive self-decomposable distributions. The innovation random variables considered in the next section are also compound Poisson and we provide the density function for the compounded variable Y explicitly.

3 Finite activity case $G(0) < \infty$

Here we focus on the particular case of self-decomposable random variables, for which the Lévy measure may be expressed as (4) with $G(0) < \infty$. We will deal with the infinite case in Section 5.

Theorem 1. *The distribution for the innovation variable W_ρ of a self-decomposable random variable X, with Lévy measure expressed as in (4), can be represented as a compound Poisson random variable*

$$W_\rho = \sum_{i=1}^{k} Y_i \tag{5}$$

where $k \sim Po(-\tau \log \rho)$. Furthermore, the compounded variable Y has density function

$$h(y) = \frac{1}{y \log \rho} \left\{ S(y/\rho) - S(y) \right\}, \tag{6}$$

where $S(y) = G(y)/G(0)$, and h has distribution function

$$H(y) = 1 + \frac{1}{\log \rho} \int_{\upsilon}^{\upsilon/\rho} \frac{S(y)}{y} dy.$$

PROOF. First note that from assumption (4) we have

$$\overline{N}_{W_\rho}(\upsilon) = N((\upsilon, \upsilon/\rho)) = \tau \int_{\upsilon}^{\upsilon/\rho} \frac{1}{y} S(y) \mathrm{d}y. \tag{7}$$

Hence the total mass for the measure N_{W_ρ} is given by

$$\begin{aligned} N_{W_\rho}((0,\infty)) &= \lim_{\upsilon \downarrow 0} \overline{N}_{W_\rho}(\upsilon) \\ &= \tau \lim_{\upsilon \to 0} \left\{ S(y) \log(y) \mid_{\upsilon}^{\upsilon/\rho} - \int_{\upsilon}^{\upsilon/\rho} \log(y) \mathrm{d}S(y) \right\} \\ &= -\tau \log \rho. \end{aligned} \tag{8}$$

Therefore, normalizing $\overline{N}_W$, we can define

$$\overline{H}(\upsilon) := \frac{\overline{N}_{W_\rho}(\upsilon)}{-\tau \log \rho} = \frac{\int_{\upsilon}^{\upsilon/\rho} \frac{1}{y} S(y) \mathrm{d}y}{-\log \rho}$$

and

$$H(\upsilon) \quad = \quad 1 + \frac{1}{\log \rho} \int_{\upsilon}^{\upsilon/\rho} \frac{1}{y} S(y) \mathrm{d}y$$

The density function, corresponding to $H(\cdot)$, is given by

$$h(y) = \frac{1}{y \log \rho} \left\{ S(y/\rho) - S(y) \right\}.$$

We now need to show that $H(\cdot)$ is a well-defined distribution function on $(0, \infty)$.

1. $h(\cdot)$ is nonnegative
2. $\overline{H}(\upsilon)$ is non-increasing since $\overline{N}'_{W_\rho}(\upsilon) = \tau \upsilon^{-1} \left\{ S(\upsilon/\rho) - S(\upsilon) \right\} < 0$
3. If $S(\cdot)$ is continuous at 0 then $H(0) = 0$
4. It remains to prove that $\overline{H}(y) \to 0$ as $y \to \infty$, that is $\overline{N}_{W_\rho}(\upsilon) \to 0$ as $\upsilon \to \infty$. For any $\epsilon > 0$ there exists $\upsilon_\epsilon > 0$, with such that $G(\upsilon) < \epsilon$ for all $\upsilon > \upsilon_\epsilon$. If $\epsilon' > 0$ and υ_ϵ as above with $\epsilon = -\epsilon' \log \rho$, then

$$\int_{\upsilon}^{\upsilon/\rho} \frac{G(x)}{x} \mathrm{d}x < \int_{\upsilon}^{\upsilon/\rho} \frac{\epsilon'}{x} \mathrm{d}x = \epsilon.$$

Once shown that $H(\cdot)$ is a well-defined distribution function we can verify that W_ρ is distributed as a compound Poisson random variable with compounded variable $Y \sim H(\cdot)$. Now

$$\mathrm{E}\left[e^{-\theta W_\rho}\right] = \mathrm{E}\left[[\mathcal{L}_Y(\theta)]^k\right] = \exp\left\{-\lambda\left(1-\int_0^\infty e^{-\theta y}\,h(y)\,\mathrm{d}y\right)\right\},$$

where $\lambda = -\tau\log\rho$, and

$$\begin{aligned}\int_0^\infty e^{-\theta y}\,h(y)\,\mathrm{d}y &= \theta\int_0^\infty e^{-\theta y}\,H(y)\,\mathrm{d}y \\ &= 1-\frac{\theta}{\lambda}\int_0^\infty e^{-\theta y}\,\overline{N}_{W_\rho}(y)\,\mathrm{d}y.\end{aligned}$$

Therefore,

$$\mathrm{E}\left[e^{-\theta W_\rho}\right] = \exp\left\{-\theta\int_0^\infty e^{-\theta y}\overline{N}_{W_\rho}(y)\mathrm{d}y\right\} \tag{9}$$

Expression (9) coincides with (3), then $H(u)$ is as stated. This completes the proof. $\square$

If $\mathrm{pr}(Y=0)=0$ then $\mathrm{pr}(W_\rho=0)=\mathrm{pr}(k=0)=e^{-\lambda}=\rho^\tau$. If $G(0)=\infty$ the representation presented in Theorem 1 is not valid, since $S(\cdot)$ is not a survival function. An example of this case is given when we assume that X is inverse Gaussian (IG). See Example 3 for more on this.

Example 1. Lawrance (1982).
If X is $\mathrm{ga}(\alpha,1)$ then $N(x,\infty)=\alpha\int_x^\infty y^{-1}\,e^{-y}\,\mathrm{d}y$ and so

$$h(y) = \frac{1}{-y\log\rho}\left\{e^{-y}-e^{-y/\rho}\right\}.$$

This has the Laplace transform given earlier in Section 1; that is,

$$\int_0^\infty e^{-\theta y}\,h(y)\,\mathrm{d}y = 1-\frac{\log\left(\frac{1+\rho\theta}{1+\theta}\right)}{\log\rho}.$$

This density generalises the exponential density which arises as $\rho\to 1$. All the moments exist and $\mathrm{E}Y^r = -(r-1)!(1-\rho^r)/\log\rho$.

Example 2.
Now let us take $G(y)=e^{-y^\xi}$, $\xi>0$. Note that for $\xi=1$ we are in the case of Example 1. The function $S(x):=G(x)$ is the survival function corresponding to a random variable $V\sim\mathrm{Weibull}(\xi,1)$. Clearly, for this case, we can verify

$$\int_0^1 un(u)\,\mathrm{d}u = \tau\int_0^1 e^{-u^\xi}\mathrm{d}u<\infty \ \text{ and } N(1,\infty)=\tau\int_1^\infty\frac{e^{-u^\xi}}{u}\mathrm{d}u<\infty,$$

leading to a valid infinitely divisible random variable X. Here the compounded random variable in the representation for W_ρ has density function

$$h(y) = \frac{1}{-y\log\rho}\left\{e^{-y^\xi}-e^{-(y/\rho)^\xi}\right\}.$$

4 Sampling Y

Let us assume that $S(\cdot)$ has a density function with respect to Lebesgue measure. Then we can write the density function $h(y)$, of the compounded random variable Y, as

$$h(y) = \frac{1}{-\log\rho}\int \frac{f(y/z)I(\rho < z < 1)}{z^2}\mathrm{d}z$$

where $f(\cdot)$ denotes the density function corresponding to $S(\cdot)$. Thus let us consider the joint density function given by

$$h(y,z) = \frac{f(y/z)I(\rho < z < 1)}{-z^2\log\rho}$$

and the marginal density function for the latent variable Z is given by

$$h(z) = \frac{I(\rho < z < 1)}{-z\log\rho}.$$

A random variable Z from this density function can be taken as $Z = \rho^U$, where U is uniform from $[0,1]$. Consequently, we can deduce that

$$Y \stackrel{\mathrm{d}}{=} V\rho^U, \tag{10}$$

where $V \sim f$ and is independent of U. Representation (10) provides us with an easy way to simulate random variates Y, and therefore also W_ρ, Figure 1 illustrates some simulations of Y corresponding to Example 1 and Example 2.

5 Infinite activity case $G(0) = \infty$

Our aim here is to approximate the distribution of the underlying innovation random variable when W_ρ is not compound Poisson; when $G(0) = \infty$. It is well known that an approximation can be made via compound Poisson random variables. In Bondesson (1982) this approximation was mentioned for general infinitely divisible Lévy processes. However, in the particular case of SD distributions, a different approximation turns out to be useful.

Our method is now introduced. In the case when $G(0) = \infty$ we can approximate (7) with

$$\overline{N}^{\epsilon}_{W_\rho}(\nu) = \int_{\nu}^{\nu/\rho}\frac{G_\epsilon(y)}{y}\mathrm{d}y$$

where

$$G_\epsilon(y) = G(\epsilon)I(y \le \epsilon) + G(y)I(y > \epsilon)$$

for $\epsilon > 0$. Here $G_\epsilon(0) = G(\epsilon) < \infty$ and $\overline{N}^{\epsilon}_{W_\rho}(\nu) \to \overline{N}_{W_\rho}(\nu)$ as $\epsilon \to 0$. We could equally use the approximation $G^*_\epsilon(y) = G(y+\epsilon)$ but in this paper we use $G_\epsilon(\cdot)$.

Proposition 1. *When $G(0) = \infty$ the approximation W^ϵ_ρ for the innovation random variable converges weakly to W_ρ, that is,*

$$W^\epsilon_\rho \stackrel{d}{\to} W_\rho, \ \text{as } \epsilon \to 0.$$

Here $\stackrel{d}{\to}$ denotes convergence in distribution.

PROOF. In order to proof convergence in distribution we use continuity theorem. Therefore, it is sufficient to show $\mathcal{L}_{W^\epsilon_\rho}(\theta) \to \mathcal{L}_{W_\rho}(\theta)$ as $\epsilon \to 0$. Using expression (3), we see that convergence of Laplace transforms is the same as proving the following;

$$\lim_{\epsilon \to 0} \int_0^\infty e^{-\theta\nu} \overline{N}^\epsilon_{W_\rho}(\nu) \mathrm{d}\nu = \int_0^\infty e^{-\theta\nu} \overline{N}_{W_\rho}(\nu) \mathrm{d}\nu.$$

Notice that $\epsilon \mapsto \overline{N}^\epsilon_{W_\rho}(\cdot)$ is a decreasing function with $\overline{N}_{W_\rho}(\cdot)$ as the limit as $\epsilon \to 0$ and therefore the monotone convergence theorem applies and the result follows. □

For the approximated Lévy measure we have

$$N^\epsilon_{W_\rho}((0,\infty)) = \lim_{\nu \downarrow 0} \overline{N}^\epsilon_{W_\rho}(\nu) = -\tau_\epsilon \log \rho \tag{11}$$

where $\tau_\epsilon = G_\epsilon(0) = G(\epsilon)$. Let us define $S_\epsilon(y) := G_\epsilon(y)/G(\epsilon)$, which is a survival function on $(0,\infty)$. Consequently, define

$$h^\epsilon(y) \quad = \quad \frac{S_\epsilon(y/\rho) - S_\epsilon(y)}{y \log \rho} \tag{12}$$

and

$$W^\epsilon_\rho = \sum_{i=1}^{k_\epsilon} Y^\epsilon_i \tag{13}$$

where $k_\epsilon \sim \mathrm{Po}(-\tau_\epsilon \log \rho)$ with the random variables Y^ϵ_i having density function given by (12). Clearly, W^ϵ_ρ has a compound Poisson distribution. The same arguments used in Theorem 1 follow in this case, leading to a well-defined probability density (12) for the compounded random variable (13). As before, we can write $Y^\epsilon \stackrel{\mathrm{d}}{=} V_\epsilon \rho^U$, where V_ϵ has distribution function $F_\epsilon(x) = 1 - S_\epsilon(x)$.

Example 3.
If $X \sim \mathrm{IG}(\delta, \gamma)$ then the corresponding Lévy measure has density $n(x) = G(x)/x$, with

$$G(x) = \frac{\delta}{\sqrt{2\pi x}} \exp\left\{-\frac{\gamma^2 x}{2}\right\},$$

and clearly $G(0) = \infty$. In this case,

$$S_\epsilon(y) = \frac{\sqrt{\epsilon}\exp\left\{-\frac{\gamma^2}{2}(y-\epsilon)\right\}}{\sqrt{y}} I(y > \epsilon) + I(y \le \epsilon)$$

defines a survival function with corresponding density function

$$f_\epsilon(y) = \frac{\sqrt{\epsilon}\exp\left\{-\frac{\gamma^2}{2}(y-\epsilon)\right\}(1+\gamma^2 y)}{2y^{3/2}} I(y > \epsilon). \tag{14}$$

The density (14) can be written as

$$f_\epsilon(x) \propto h(x)k(x)I(x > 0), \qquad x = y - \epsilon \tag{15}$$

and

$$h(x) = \frac{\sqrt{\epsilon}}{2(x+\epsilon)^{3/2}}, \quad k(x) = e^{-\gamma^2 x/2}\left\{1+\gamma^2(x+\epsilon)\right\}. \tag{16}$$

Hence, in order to simulate from the random variable V_ϵ with density (14) we can simulate from (15) and add ϵ. The decomposition in (15) allows us to use the acceptance-rejection method by simulating from $h(\cdot)$ in (16) and with acceptance criteria $U \leq k(x)/M$, $M = \sup_x k(x) = k(m)$ with $m = \max\{\gamma^{-2} - \epsilon, 0\}$ and U is an uniform $[0,1]$ random variate. See Rubinstein (1981). Therefore, to simulate from the random variable (13) we follow the next steps:

- For any $\epsilon > 0$ simulate $k_\epsilon \sim \text{Po}(-G(\epsilon)\log\rho)$.
- Simulate k_ϵ independent random numbers from an uniform distribution in $[0,1]$ and k_ϵ independent random numbers from V_ϵ as described above.
- Compute (13).

In order to illustrate this method graphically, notice that we can approximate a SD random variate X by simulating from $X^\epsilon = \rho X + W_\rho^\epsilon$. If $X \sim \text{IG}(\delta,\gamma)$ then $\rho X \sim \text{IG}(\delta\sqrt{\rho}, \gamma/\sqrt{\rho})$, therefore, a random variate from X can be approximated as the sum of a random number from $\rho X \sim \text{IG}(\delta\sqrt{\rho}, \gamma/\sqrt{\rho})$ and a random number from W_ρ^ϵ.

As an alternative to our direct sampling method, V_ϵ can be simulated using the Inverse cdf method. This corresponds to the standard inverse Lévy method. Figure 2 reports some simulations using the inverse cdf method and our method. The reduction in computer-time is considerable; the simulation of 3,000 random variables took 0.36 seconds for the inverse cdf method compared with 0.06 seconds for ours. All computations were done in Ox; see Doornik (2001).

6 Relation with OU type processes and their representations

In Barndorff-Nielsen and Shephard (2001) a crucial feature is the simulation of the innovations for OU type processes. OU type processes are stationary processes with marginal distributions given by SD distributions (see for example Sato, 1999). Here we focus on OU type processes, when a prior choice of the stationary distribution is given. An OU process can be represented as follows

$$X(t) \quad = \quad \rho^t X(0) + \rho^t \int_0^{at} e^s \mathrm{d}L(s) \quad \text{for } \rho = e^{-a}, a > 0 \tag{17}$$

where $L(\cdot)$ is a Lévy process on $(0,\infty)$. See Wolfe (1982). In order to simulate from the innovation part (the second term in (17)), Barndorff-Nielsen and Shephard (2001) made use of the following result;

$$\int_0^{\Upsilon} f(s)\mathrm{d}L(s) \stackrel{\mathrm{d}}{=} \sum_{i=1}^{\infty} M^{-1}(\xi_i/\Upsilon) f(\Upsilon r_i)$$

where M^{-1} denotes the inverse of the tail Lévy measure corresponding to $L(1)$. Here $\{\xi_i\}$ and $\{r_i\}$ are two independent sequences of random variables, with the r_i being independent and identically distributed from the uniform distribution on $[0,1]$ and $\xi_1 < \cdots < \xi_i < \cdots$ are the jump times of a Poisson process with intensity 1. It is worth noting that the above result can be seen as particular case of a representation given by Ferguson and Klass (1972) (see Walker, 2001). For the OU-innovation this representation simplifies as

$$\begin{aligned} X(t) &= \rho^t X(0) + \rho^t \int_0^{at} e^s \mathrm{d}L(s) \\ &\stackrel{\mathrm{d}}{=} \rho^t X(0) + \sum_{i=1}^{\infty} \rho^{(1-r_i)t} M^{-1}(\xi_i/at) \\ &\stackrel{\mathrm{d}}{=} \rho^t X(0) + \sum_{i=1}^{\infty} \rho^{tr_i} M^{-1}(\xi_i/at). \end{aligned} \tag{18}$$

Example 4.
Consider a Lévy process L with gamma $\mathrm{ga}(\tau, 1)$ increments, thus $M^{-1}(x) = \max\{0, -\log(x/\tau)\}$. In this case the innovation part (the second summand in (18)) is represented as

$$\begin{aligned} \rho^t \int_0^{at} e^s \mathrm{d}L(s) &\stackrel{\mathrm{d}}{=} \sum_{i=1}^{\infty} \rho^{tr_i} \log(1/c_i) I(0 < c_i < 1) \\ &= \sum_{i=1}^{N(1)} \rho^{tr_i} \log(1/c_i) \end{aligned}$$

where $c_1 < \cdots < c_i < \cdots$ are the jump times of a Poisson process with intensity $a\tau t$ (or $-t\tau\log(\rho)$) and $N(1)$ corresponding number of jumps before 1. Let k be the number of jumps before 1, then given k the $\{c_i\}$ are independent and identically distributed from a uniform distribution in $[0, 1]$. Hence if we define $V_i = -\log(c_i)$ thus $V_i \sim \mathrm{Exp}(1)$ and therefore

$$\rho^t \int_0^{at} e^s \mathrm{d}L(s) \stackrel{\mathrm{d}}{=} \sum_{i=1}^{N(1)} \rho^{tr_i} V_i$$

which, for $t = 1$, is exactly the representation provided in Example 1.

Knowledge of the distribution for the innovation random variable (or an approximation to it) of a self-decomposable random variable gives new ways of simulating the innovation part of an OU type process. The point of view presented here allows us to use any suitable random variable simulation method (not only inverse methods).

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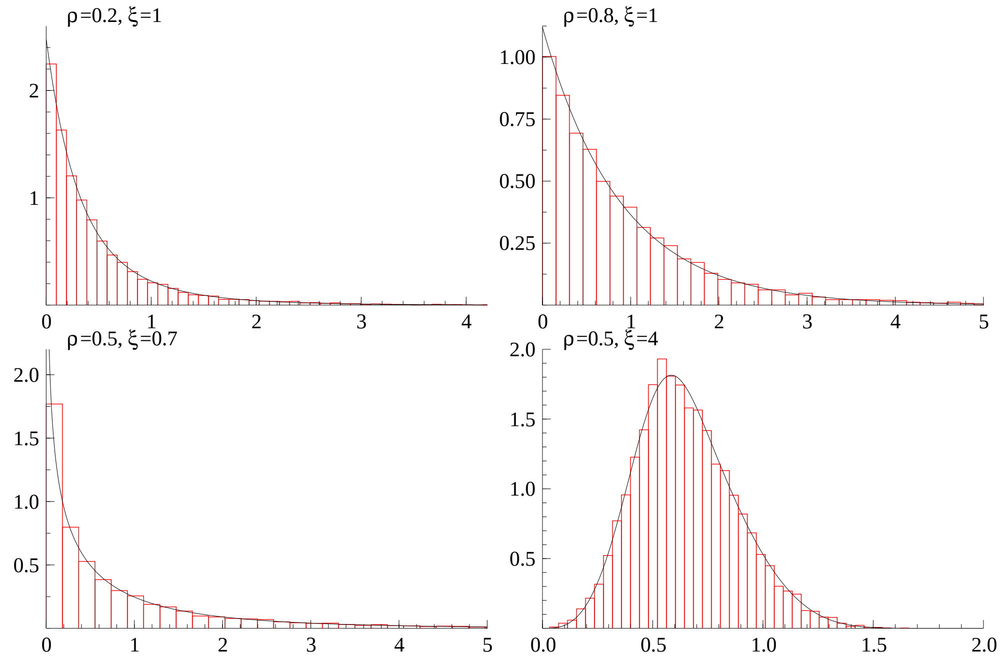

Fig. 1: Simulations from the compounded random variable Y, using representation (10) with $V \sim \text{Weibull}(\xi, 1)$. $S(y) = e^{-y^{\xi}}$, 10,000 simulations. The solid lines represent the true densities.

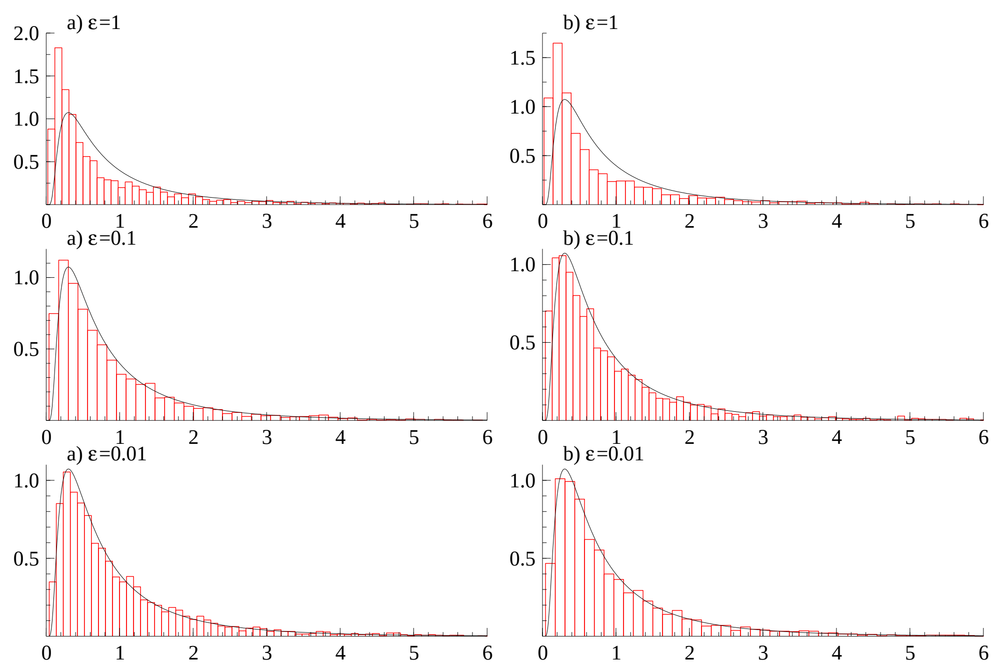

Fig. 2: Approximation of self-decomposable random variable IG(1,1) using 3000 simulations of W_{ρ}^{ϵ} and ρX with $\rho = 0.5$. The simulations for the compounded random variable were done using the inverse CDF method (a) and the acceptance-rejection method (b). The solid line represents the true density.